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ORIGINAL ARTICLE

ANALYTICAL SOLUTION OF THE SECOND ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT: One, or two, solutions of $y'' + p(x)y' + Q(x)y = 0$, where $p(x)$ and $Q(x)$ are continuous functions, are provided by $y = ce^{\int H(x)dx}$ (where, c is an arbitrary constant), and $H(x)$is obtained from the HR system as following:

$$HR\begin{cases} R(x) - H(x) = p(x) \\ R'(x) + R(x)H(x) = p'(x) - Q(x) \end{cases}$$

In case of arriving at one solution of the differential equation, the second one would be found by

$$y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx.$$

Two methods are provided for the solution of HR system. In the first method, $R(x)$ is guessed and then $H(x)$is computed. In the second method

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D + z(x)$$

Where; $A, B, K, C, G, S, D,$ and $z(x)$ are unknown constant.
In case $p(x) = 0$, we set $A = C = G = S = 0$, and if $Q(x) = 0$ we set $B = K = C = G = 0$. Then, assuming $z(x) = 0$, we substitute the equivalent of $R(x)$ into the system. In case of arriving at a value for the unknown constants through solving the system, the first assumption $z(x) = 0$ is satisfied and $H(x)$ can be computed; otherwise, $z(x) \neq 0$. The method for obtaining $z(x) \neq 0$ can be based on guessing.
KEYWORDS: Analytical Solution, Homogeneous, Linear, Differential Equations.

INTRODUCTION

In the differential Equation 2:

$$y'' + p(x)y' + Q(x)y = r(x) \qquad (2)$$

If $r(x) \neq 0$ and $r(x) = 0$, the differential equation is, in turn, called nonhomogeneous and homogenous (Birkhoff and Rota, 1989). In homogeneous equation, y_1 and y_2, are linearly independent solutions. Therefore, the general solution is $y_h = c_1 y_1 + c_2 y_2$, (where, c_1 and c_2 are constants). To find y_h, more common parameter changing method can be employed. In this technique, y_1 is guessed and $y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx$. If y_p is one simple solution of the nonhomogeneous Equation 2, the general answer of the nonhomogeneous equation is y_h plus y_p. For computing y_p, in the presence of y_h, parameter changing method can be used. Therefore, in the presence y_1 , y_h can be obtained, and in case the differential Equation 2 is nonhomogeneous, y_p is computed through parameter changing method, and finally the general answer of the nonhomogeneous equation would be provided. Currently, there is no specific generally applicable analytical method for arriving at general answer of Equation 2 by merely computing y_1. Therefore, the attempts aimed to develop a method for calculation of y_1 (Burkill, 1959; Boyce and DiPrima, 2005).

HR SYSTEM

One, or two, solutions of $y'' + p(x)y' + Q(x)y = 0$, where $p(x)$ and $Q(x)$ are continuous functions, are provided by $y = ce^{\int H(x)dx}$ (where, c is an arbitrary constant), and $H(x)$ is obtained from the HR system as following:

$$HR\begin{cases} R(x) - H(x) = p(x) \\ R'(x) + R(x)H(x) = p'(x) - Q(x) \end{cases}$$

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In case of arriving at one solution of the differential equation, the second one would be $y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx$.

$$y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx.$$

2.1. Proof

In Differential Equation 1:

$$y'' + p(x)y' + Q(x)y = 0 \qquad (1)$$

Assuming $y \neq 0$

$$\frac{y''}{y} + p(x)\frac{y'}{y} + Q(x) = 0 \Rightarrow \frac{y''y - y'^2 + y'^2}{y^2} + p(x)\frac{y'}{y} + Q(x) = 0$$

$$\Rightarrow (\frac{y'}{y})' + (\frac{y'}{y})^2 + p(x)\frac{y'}{y} + Q(x) = 0$$

And, assuming $H(x) = \frac{y'}{y}$, then

$$H'(x) + H^2(x) + p(x)H(x) + Q(x) = 0 \Rightarrow$$

$$H'(x) + H^2(x) + p(x)H(x) = -Q(x) \qquad (3)$$

We define

$$R(x) = p(x) + H(x)$$

As a result

$$H(x)' = R(x)' - p(x)' \quad , \qquad p(x) = R(x) - H(x) \qquad (4)$$

Now, we substitute the equivalents of $p(x)$ and $H(x)$ into Equation 3

$$R(x)' - p(x)' + H^2(x) + \big(R(x) - H(x)\big)H(x) = -Q(x) \Rightarrow$$

$$R'(x) + R(x)H(x) = p'(x) - Q(x) \qquad (5)$$

We define HR system by means of Equation 4 and 5

$$HR \begin{cases} R(x) - H(x) = p(x) \\ R'(x) + R(x)H(x) = p'(x) - Q(x) \end{cases}$$

Also

$$H(x) = \frac{y'}{y} \Rightarrow y' - H(x)y = 0 \qquad (6)$$

We multiply Equation 6 by integral factor $\frac{1}{e^{\int H(x)\,dx}}$

$$y'\frac{1}{e^{\int H(x)\,dx}} - y\frac{H(x)}{e^{\int H(x)\,dx}} = 0 \Rightarrow (\frac{y}{e^{\int H(x)\,dx}})' = 0 \Rightarrow y = ce^{\int H(x)dx}$$

$y = ce^{\int H(x)dx}$ is the solution of Equation 1, (where, c is the arbitrary constant). In case $H(x)_1$ is arrived at by solving HR system, employing $c = 1$, one solution of differential equation would be $y_1 = e^{\int H_1(x)dx}$. Using parameter changing method, we have:

2.2. Solution for the HR system

The HR system is solved by two methods. In the first method, $R(x)$ is guessed using $R(x) - H(x) = p(x)$ and $R'(x) + R(x)H(x) = p'(x) - Q(x)$, and then $H(x)$ is calculated. In the second method,

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D + z(x) \qquad (7)$$

Where; $A, B, K, C, G, S, D,$ and $z(x)$ are taken as unknown constants. In case $p(x) = 0$ and $Q(x) = 0$, we set $A = C = G = S = 0$ and $B = K = C = G = 0$, respectively. Assuming $z(x) = 0$, we substitute equivalent of $R(x)$ into the system and arrive at a solution, if any, for the constants. Finally, $H(x)$ is computed. In case of no answer for the constants, the assumption $z(x) = 0$ is not satisfied. The first method can be used for arriving at $z(x) \neq 0$. From now on, since Equation 7 is used with the assumption $z(x) = 0$, so for factoring, we remove $z(x)$ from the right side of Equation 7.

2.3. proof

The HR system of the differential equation $y'' + p(x)y' + Q(x)y = 0$ is as follow

$$HR \begin{cases} R(x) - H(x) = p(x) \\ R'(x) + R(x)H(x) = p'(x) - Q(x) \end{cases}$$

In order to solve the system, it is tried to choose $R(x)$ in appropriate form. Using the first method for solving the system, $R(x)$ can equate to $A'p(x)$, $A'p(x) + BQ(x)$,$A'p(x) + \sqrt{KQ(x)}$, $A'p(x) + C\frac{Q(x)}{p(x)}$, $A'p(x) + G\frac{p(x)}{Q(x)}$, $A'p(x) + \frac{S}{p(x)}$, and $A'p(x) + D$ where $A', B, K, C, G, S,$ and D are constants. Now, to control all possible guesses and etc., we consider a set of guesses which are in following form:

$$R(x) = 7A'p(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D$$

Generally, $R(x)$ can have the following form

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D + z(x) \qquad (7)$$

Where, $A, B, K, C, G, S, D,$ and $z(x)$ are unknown constants.

HR SYSTEM SOLUTION OF THE DIFFERENTIAL EQUATION $y'' + ay' + by = 0$

First, we prove Equation 8

$$c_1e^{iu} + c_2e^{-iu} = A\cos u + B\sin u \qquad (8)$$

Based on Euler's formula

$e^{i\theta} = \cos\theta + i\sin\theta$

By choosing

$$c_1 = \frac{1}{2}(\mathrm{A} - i\mathrm{B}) \quad , \quad c_2 = \frac{1}{2}(\mathrm{A} + i\mathrm{B})$$

And using Euler's formula and substituting it by the left side of Equation 8, we have:

$$\frac{(\mathrm{A} - i\mathrm{B})}{2}(\cos u + i\sin u) + \frac{(\mathrm{A} + i\mathrm{B})}{2}(\cos u - i\sin u) = A\cos u + B\sin u$$

The right side of the equation would be obtained after simplification.
If $H(x)_1 = v + iu$ and $H(x)_2 = v - iu$ are obtained by solving the HR system, the general solution of the differential equation would be

$$y_h = c_1 e^{\int H(x)_1 dx} + c_2 e^{\int H(x)_2 dx} = c_1 e^{\int v+iu\, dx} + c_2 e^{\int v-iu\, dx}$$
$$= e^{\int v dx}(c_1 e^{i\int u dx} + c_2 e^{-i\int u dx})$$

Using Equation 8, we get

$$y_h = e^{\int v dx}(A\cos\int u\, dx + B\sin\int u\, dx) \tag{9}$$

From now on, when $H(x) = v \pm iu$ is obtained, we use Equation 9 to compute y_h. In order to solve the HR system of the differential equation $y'' + ay' + by = 0$, by the second method, we have:

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D =$$
$$Aa + Bb + \sqrt{Kb} + C\frac{b}{a} + G\frac{a}{b} + \frac{S}{a} + D$$

By changing variable, we set $k = Aa + Bb + \sqrt{Kb} + C\frac{b}{a} + G\frac{a}{b} + \frac{S}{a} + D$ (where, k is a constant), we have:

$$HR\begin{cases} R(x) - H(x) = a \Rightarrow R(x) = H(x) + a \\ R'(x) + R(x)H(x) = -b \Rightarrow 0 + (H(x) + a\,)H(x) = -b \\ \Rightarrow H(x)^2 + aH(x) + b = 0 \end{cases} \tag{10}$$

Given the expression $R(x) - H(x) = a$, if $R(x)$ was a constant, $H(x)$ would be a constant too. Solving the second order Equation 10, which is the characteristic equation, a constant value is obtained for $H(x)$, so the assumption $R(x) = k$ is satisfied. As a result, for solving the HR system of second order homogeneous differential equation with constant coefficients, $R(x)$ is a constant value. If the second order Equation 10 has two distinct real roots $H(x) = H_1, H_2$, then:

$$y_h = c_1 e^{\int H_1 dx} + c_2 e^{\int H_2 dx} = c_1 e^{H_1 x} + c_2 e^{H_2 x}$$

If second order Equation 10 has two distinct complex roots $H(x) = p \pm iq$, then we have

$$y_h = e^{\int p dx}(A\cos\int q\, dx + B\sin\int q\, dx) = e^{px}(A\cos qx + B\sin qx)$$

If the second order Equation 10 has a double root, we would have $H(x) = -\frac{a}{2}$ and

$$y_1 = c_1 e^{\int -\frac{a}{2}dx} = c_1 e^{-\frac{a}{2}x}$$

By setting $c_1 = 1$, $y_1 = e^{-\frac{a}{2}x}$ would be the solution of the differential equation. As a result

$$y_2 = y_1\int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx = e^{-\frac{a}{2}x}\int \frac{1}{e^{-ax}} e^{-\int a\, dx}\, dx = xe^{-\frac{a}{2}x}$$

And

$$y_h = c_1 e^{-\frac{a}{2}x} + c_2 x e^{-\frac{a}{2}x} = (c_1 + c_2 x)e^{-\frac{a}{2}x}$$

HR SYSTEM SOLUTION OF THE DIFFERENTIAL EQUATION $y'' + (ax+b)y' + cy = 0, (a \neq 0, c \neq 0)$

In order to solve the HR system of differential Equation 11

$$y'' + (ax + b)y' + cy = 0 \tag{11}$$

assuming $c \neq 0$, $a \neq 0$, and $b = 0$, we have:

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D =$$
$$A(ax) + Bc + \sqrt{Kc} + C\frac{c}{ax} + G\frac{ax}{c} + \frac{S}{ax} + D =$$
$$(Aa + G\frac{a}{c})x + \frac{Cc + S}{a}\frac{1}{x} + Bc + \sqrt{Kc} + D$$

Given $Bc + \sqrt{Kc} + D = F$, $\frac{Cc+S}{a} = V$, and $Aa + G\frac{a}{c} = L$, we have

$$R(x) = Lx + \frac{V}{x} + F$$

$$HR\begin{cases} R(x) - H(x) = ax \Rightarrow H(x) = R(x) - ax = (L - a)x + \frac{V}{x} + F \\ R'(x) + R(x)H(x) = a - c \\ \Rightarrow L - \frac{V}{x^2} + (Lx + \frac{V}{x} + F)((L - a)x + \frac{V}{x} + F) \equiv a - c \\ \Rightarrow \begin{cases} L + LV + V(L - a) + F^2 = a - c \\ (V^2 - V)\frac{1}{x^2} = 0 \\ L(L - a)x^2 = 0 \\ \big(LF + F(L - a)\big)x = 0 \\ (FV + VF)\frac{1}{x} = 0 \end{cases} \end{cases}$$

Given the primary assumption $c \neq 0$, $a \neq 0$, and $b = 0$, if $a = c$ the solution would be:
$V = L = F = 0$. As a result

$$y_1 = c_1 e^{\int H(x)dx} = c_1 e^{\int -ax\, dx} = c_1 e^{-\frac{a}{2}x^2}$$

By setting $c_1 = 1$, $y_1 = e^{-\frac{a}{2}x^2}$ would be the solution of the differential equation. As a result

$$y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx = e^{-\frac{a}{2}x^2} \int \frac{1}{e^{-ax^2}} e^{-\int ax\, dx}\, dx = e^{-\frac{a}{2}x^2} \int e^{\frac{a}{2}x^2}\, dx$$

And

$$y_h = c_1 e^{-\frac{a}{2}x^2} + c_2 e^{-\frac{a}{2}x^2} \int e^{\frac{a}{2}x^2}\, dx$$

If $a = -c$, the solution would be $F = 0$, $L = a$, and $V = 1$. As a result

$$y_1 = c_1 e^{\int H(x)dx} = c_1 e^{\int \frac{1}{x} dx} = c_1 x$$

By setting $c_1 = 1$, $y_1 = x$ would be the solution of the differential equation. As a result

$$y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx = x \int \frac{1}{x^2} e^{-\int ax\, dx}\, dx = x \int \frac{1}{x^2} e^{-\frac{a}{2}x^2}\, dx$$

And

$$y_h = c_1 x + c_2 x \int \frac{1}{x^2} e^{-\frac{a}{2}x^2}\, dx$$

If $2a = c$, $L = F = 0$ and $V = 1$ would be the solution. As a result

$$y_1 = c_1 e^{\int H(x)dx} = c_1 e^{\int -ax + \frac{1}{x} dx} = c_1 x e^{-\frac{a}{2}x^2}$$

By setting $c_1 = 1$, the solution of the differential equation would be $y_1 = xe^{-\frac{a}{2}x^2}$, so we would have

$$y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx = xe^{-\frac{a}{2}x^2} \int \frac{1}{x^2 e^{-ax^2}} e^{-\int ax\, dx}\, dx = xe^{-\frac{a}{2}x^2} \int \frac{1}{x^2} e^{\frac{a}{2}x^2}\, dx$$

And

$$y_h = xe^{-\frac{a}{2}x^2} \left(c_1 + c_2 \int \frac{1}{x^2} e^{\frac{a}{2}x^2}\, dx\right)$$

Assuming $c \neq 0$, $b \neq 0$, and $a \neq 0$, and using the second method we would have

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D =$$
$$A(ax+b) + Bc + \sqrt{Kc} + C\frac{c}{ax+b} + G\frac{ax+b}{c} + \frac{S}{ax+b} + D =$$
$$A(ax+b) + \frac{Cc+S}{ax+b} + Bc + \sqrt{Kc} + G\frac{a}{c}x + G\frac{b}{c} + D$$

Given $Bc + \sqrt{Kc} + G\frac{b}{c} + D = F$, $Cc + S = V$, and $G\frac{a}{c} = L$, we have:

$$R(x) = A(ax+b) + \frac{V}{ax+b} + Lx + F$$

$$HR \begin{cases} R(x) - H(x) = ax + b \Rightarrow H(x) = R(x) - ax - b = (A-1)(ax+b) + \frac{V}{ax+b} + Lx + F \\ R'(x) + R(x)H(x) = a - c \Rightarrow Aa - \frac{Va}{(ax+b)^2} + L \\ \quad +\left(A(ax+b) + \frac{V}{ax+b} + Lx + F\right)\left((A-1)(ax+b) + \frac{V}{ax+b} + Lx + F\right) \equiv a - c \\ \Rightarrow \begin{cases} Aa + L + 2AV + F^2 - V = a - c \\ (V^2 - Va)\frac{1}{(ax+b)^2} = 0 \\ (A^2 - A)(ax+b)^2 = 0 \\ (2AF - F)(ax+b) = 0 \\ (2AL - L)x(ax+b) = 0 \\ (2LV)\frac{x}{ax+b} = 0 \\ (2FV)\frac{1}{ax+b} = 0 \\ L^2x^2 = 0 \\ (2FL)x = 0 \end{cases} \end{cases}$$

Given the primary assumption $c \neq 0$, $b \neq 0$, and $a \neq 0$,the solution would be $L = F = A = V = 0$, if $a = c$. As a result

$$y_1 = c_1 e^{\int H(x)dx} = c_1 e^{\int -ax - b\, dx} = c_1 e^{-\frac{a}{2}x^2 - bx}$$

If $a = -c$, the solution would be $A = 1$, ,$V = a$, and $L = F = 0$. As a result

$$y_1 = c_1 e^{\int H(x)dx} = c_1 e^{\int \frac{a}{ax+b} dx} = c_1(ax+b)$$

If $2a = c$, the solution would be $V = a$, and $L = F = A = 0$ As a result

$$y_1 = c_1 e^{\int H(x)dx} = c_1 e^{\int \frac{a}{ax+b} - ax - b\, dx} = c_1(ax+b)e^{-\frac{a}{2}x^2 - bx}$$

In this case, due to the simplicity of the problem, calculation of the second and general solutions would be upon the reader.

EXAMPLES

5.1. Example 1

Solve the following HR system of the differential equation using the first method.

$$y'' - y' - (2e^x + e^{2x})y = 0$$

Solution

We set up the HR system

$$HR \begin{cases} R(x) - H(x) = -1 \\ R'(x) + R(x)H(x) = 2e^x + e^{2x} \end{cases}$$

Given $R(x) - H(x) = -1$, as the right side is constant, we first assume that $H(x)$ and $R(x)$ are equal in terms of form, i.e. both of them are exponential, polynomial, sine, or etc. Since the difference between $H(x)$ and $R(x)$ is one, so the expression $R'(x) + R(x)H(x)$ approximately equates to the sum of a function derivative with its quadratic. On the right side the equation $R'(x) + R(x)H(x) = 2e^x + e^{2x}$, function's derivative e^x and its quadratic e^{2x}, are seen, regardless of their coefficients. Therefore, the assumption

$R(x) = ke^x + d$ is reasonable (where, d and k are constants). With this assumption, we solve HR system

$$HR\begin{cases} R(x) - H(x) = -1 \Rightarrow H(x) = ke^x + d + 1 \\ R'(x) + R(x)H(x) = 2e^x + e^{2x} \Rightarrow ke^x + (ke^x + d)(ke^x + d + 1) \equiv 2e^x + e^{2x} \\ \Rightarrow ke^x + k^2e^{2x} + kde^x + ke^x + kde^x + d^2 + d \equiv 2e^x + e^{2x} \\ \Rightarrow \begin{cases} 2k(1+d)e^x = 2e^x \\ k^2e^{2x} = e^{2x} \\ d^2 + d = 0 \end{cases} \Rightarrow \begin{matrix} k = 1 \\ d = 0 \end{matrix} \Rightarrow H(x) = e^x + 1 \end{cases}$$

$$y_1 = c_1 e^{\int H(x)_1 dx} = c_1 e^{\int e^x + 1 dx} = c_1 e^{e^x + x}$$

By setting $c_1 = 1$, $y_1 = e^{e^x + x}$ would be the solution of the differential equation. As a result

$$y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p(x)dx}\, dx = e^{e^x+x} \int \frac{1}{e^{2e^x+2x}} e^{\int dx}\, dx = e^{e^x+x} \int e^{-2e^x - x}\, dx$$

And

$$y_h = c_1 e^{e^x+x} + c_2 e^{e^x+x} \int e^{-2e^x-x}\, dx$$

5.2. Example 2

$$y'' - \frac{2x+3}{x+1} y' + \frac{x+2}{x+1} y = 0$$

Solution

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D =$$
$$A(-2 - \frac{1}{x+1}) + B(1 + \frac{1}{x+1}) + \sqrt{K(1 + \frac{1}{x+1})} - C\frac{x+2}{2x+3} - G\frac{2x+3}{x+2} - S\frac{x+1}{2x+3} + D =$$
$$(B-A)\frac{1}{x+1} + \sqrt{K(1 + \frac{1}{x+1})} + \frac{S-C}{2}\frac{1}{2x+3} + \frac{G}{x+2} - 2A + B - 2G - \frac{C+S}{2} + D$$

Given $B - A = a$, $\frac{S-C}{2} = c$, and $-2A + B - 2G - \frac{C+S}{2} + D = d$, then we would have

$$R(x) = \frac{a}{x+1} + \sqrt{K(1 + \frac{1}{x+1})} + \frac{c}{2x+3} + \frac{G}{x+2} + d$$

Given *HR system*

$$HR\begin{cases} R(x) - H(x) = -2 - \dfrac{1}{x+1} \\ R'(x) + R(x)H(x) = \dfrac{1}{(x+1)^2} - \dfrac{1}{x+1} - 1 \end{cases}$$

Derivative of the expressions $\sqrt{K(1 + \frac{1}{x+1})}$, $\frac{c}{2x+3}$, and $\frac{G}{x+2}$ does not exist on the right side of equation $R'(x) + R(x)H(x) = \frac{1}{(x+1)^2} - \frac{1}{x+1} - 1$. So, we guess that $K = c = G = 0$ would be arrived at by solving the system. Therefore, we set $R(x) = \frac{a}{x+1} + d$

$$HR\begin{cases} R(x) - H(x) = -2 - \dfrac{1}{x+1} \Rightarrow H(x) = \dfrac{a+1}{x+1} + d + 2 \\ R'(x) + R(x)H(x) = -1 - \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} \Rightarrow \\ \dfrac{-a}{(x+1)^2} + (\dfrac{a}{x+1} + d)(\dfrac{a+1}{x+1} + d + 2) \equiv -1 - \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} \Rightarrow \\ \dfrac{a^2}{(x+1)^2} + \dfrac{1}{x+1}(2ad + 2a + d) + d^2 + 2d \equiv -1 - \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} \\ \Rightarrow \begin{cases} \dfrac{a^2}{(x+1)^2} = \dfrac{1}{(x+1)^2} \\ \dfrac{1}{x+1}(2ad + 2a + d) = -\dfrac{1}{x+1} \\ d^2 + 2d = -1 \end{cases} \Rightarrow \begin{matrix} d = -1 \\ a = \pm 1 \end{matrix} \Rightarrow H(x) = \dfrac{2}{x+1} + 1 \text{ and } 1 \end{cases}$$

$$y_h = c_1 e^{\int H_1(x)dx} + c_2 e^{\int H_2(x)dx} = c_1 e^{\int \frac{2}{x+1} + 1 dx} + c_2 e^{\int dx} = c_1 (x+1)^2 e^x + c_2 e^x$$

5.3. Example 3

$x^2y'' + xy' + (x^2 - v^2)y = 0$ (Bessel Equation)

Solution

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D =$$
$$A\frac{1}{x} + B(1 - \frac{v^2}{x^2}) + \sqrt{K(1 - \frac{v^2}{x^2})} + C(x - \frac{v^2}{x}) + G(\frac{x}{x^2 - v^2}) + Sx + D =$$
$$(A - Cv^2)\frac{1}{x} - Bv^2\frac{1}{x^2} + \sqrt{K(1 - \frac{v^2}{x^2})} + (C+S)x + G(\frac{x}{x^2 - v^2}) + B + D$$

Given $-Cv^2 = a$, $-Bv^2 = b$, $C + S = c$, and $B + D = d$, then

$$R(x) = \frac{a}{x} + \frac{b}{x^2} + \sqrt{K(1 - \frac{v^2}{x^2})} + cx + G(\frac{x}{x^2 - v^2}) + d$$

Given HR system

$$HR\begin{cases} R(x) - H(x) = \dfrac{1}{x} \\ R'(x) + R(x)H(x) = \dfrac{v^2 - 1}{x^2} - 1 \end{cases}$$

Derivatives of the expressions $\frac{b}{x^2}$, $\sqrt{K(1 - \frac{v^2}{x^2})}$, $G(\frac{x}{x^2 - v^2})$, and the expression c^2x^2, which would be seen in the result of $R(x)H(x)$, do not exist on the right side of the equation $R'(x) + R(x)H(x) = \frac{v^2-1}{x^2} - 1$. So, we guess we would find $b = K = c = G = 0$ by solving the system. Therefore, we set $R(x) = \frac{a}{x} + d$

$$HR\begin{cases} R(x)-H(x)=\frac{1}{x}\Rightarrow H(x)=\frac{a-1}{x}+d \\ R'(x)+R(x)H(x)=\frac{v^2-1}{x^2}-1\Rightarrow -\frac{a}{x^2}+(\frac{a}{x}+d)(\frac{a-1}{x}+d)\equiv\frac{v^2-1}{x^2}-1 \\ \Rightarrow\frac{(a^2-2a)}{x^2}+\frac{(2ad-d)}{x}+d^2\equiv\frac{v^2-1}{x^2}-1 \\ \Rightarrow\begin{cases}\frac{(a^2-2a)}{x^2}=\frac{v^2-1}{x^2} \\ \frac{(2ad-d)}{x}=0 \\ d^2=-1\end{cases}\Rightarrow\begin{matrix} v=\pm\frac{1}{2} \\ a=\frac{1}{2} \\ d=\pm i\end{matrix}\Rightarrow H(x)=\pm i-\frac{1}{2x}\end{cases}$$

$$y_h=e^{\int vdx}(A\cos\int u\,dx+B\sin\int u\,dx)=e^{\int-\frac{1}{2x}dx}(A\cos\int dx+B\sin\int dx)=$$
$$\frac{1}{\sqrt{x}}(Acosx+Bsinx)$$

The equation $R(x)=\frac{a}{x}+d$ is just answerable when $v=\pm\frac{1}{2}$. For other values of v, another form of $R(x)$ is required.

5.4. Example 4

$$y''-2\tan x\,y'-10y=0$$

Solution

$$R(x)=Ap(x)+BQ(x)+\sqrt{KQ(x)}+C\frac{Q(x)}{p(x)}+G\frac{p(x)}{Q(x)}+\frac{S}{p(x)}+D=$$
$$-2A\tan x-10B+\sqrt{-10K}+C\frac{10}{2\tan x}+G\frac{2\tan x}{10}-\frac{S}{2\tan x}+D=$$
$$(-2A+G\frac{1}{5})\tan x+(5C-\frac{S}{2})\frac{1}{\tan x}+\sqrt{-10K}-10B+D$$

Given $-2A+G\frac{1}{5}=a$, $5C-\frac{S}{2}=b$, and $\sqrt{-10K}-10B+D=d$, then

$$R(x)=a\tan x+\frac{b}{\tan x}+d$$

Given HR system

$$HR\begin{cases}R(x)-H(x)=-2\tan x \\ R'(x)+R(x)H(x)=8-2tan^2x\end{cases}$$

The derivative of $\frac{b}{\tan x}$ does not exist on the right side of the equation $R'(x)+R(x)H(x)=8-2tan^2x$. Therefore, we guess that $b=0$ would be obtained by solving the system, so we set $R(x)=a\tan x+d$.

$$HR\begin{cases}R(x)-H(x)=-2\tan x\Rightarrow H(x)=(a+2)\tan x+d \\ R'(x)+R(x)H(x)=8-2tan^2x \\ \Rightarrow a(tan^2x+1)+((a+2)\tan x+d)(a\tan x+d)\equiv 8-2tan^2x \\ \Rightarrow(a^2+3a)tan^2x+(2ad+2d)\tan x+d^2+a\equiv 8-2tan^2x \\ \Rightarrow\begin{cases}(a^2+3a)tan^2x=-2tan^2x \\ (2ad+2d)\tan x=0 \\ d^2+a=8\end{cases}\Rightarrow\begin{matrix}a=-1 \\ d=\pm3\end{matrix}\Rightarrow H(x)=\tan x\pm3\end{cases}$$

$$y_h=c_1e^{\int\tan x+3dx}+c_2e^{\int\tan x-3dx}=c_1e^{-lncosx+3x}+c_2e^{-lncosx-3x}$$
$$=\frac{1}{cosx}(c_1e^{3x}+c_2e^{-3x})$$

5.5. Example 5

$$y''-y'+e^{2x}y=0$$

Solution

$$R(x)=Ap(x)+BQ(x)+\sqrt{KQ(x)}+C\frac{Q(x)}{p(x)}+G\frac{p(x)}{Q(x)}+\frac{S}{p(x)}+D=$$
$$-A+Be^{2x}+\sqrt{Ke^{2x}}-Ce^{2x}-G\frac{1}{e^{2x}}-S+D=(B-C)e^{2x}+\sqrt{K}e^x-G\frac{1}{e^{2x}}-S+D-A$$

Given $-C=b$, $\sqrt{K}=k$, $-G=g$, and $-S+D-A=d$, then

$$R(x)=be^{2x}+ke^x+g\frac{1}{e^{2x}}+d$$

Given HR system

$$HR\begin{cases}R(x)-H(x)=-1 \\ R'(x)+R(x)H(x)=-e^{2x}\end{cases}$$

The derivative of $ke^x, g\frac{1}{e^{2x}}$ does not exist on the right side of the equation $R'(x)+R(x)H(x)=-e^{2x}$. So we guess that $k=g=0$ would be obtained by solving the system. Therefore, $R(x)=be^{2x}+d$ was set. But, we did not arrive at a solution with this assumption. So, we triedassuming $R(x)=ke^x+d$

$$HR\begin{cases}R(x)-H(x)=-1\Rightarrow H(x)=ke^x+d+1 \\ R'(x)+R(x)H(x)=-e^{2x}\Rightarrow ke^x+(ke^x+d)(ke^x+d+1)\equiv -e^{2x} \\ \Rightarrow ke^x+k^2e^{2x}+kde^x+ke^x+kde^x+d^2+d\equiv -e^{2x} \\ \Rightarrow\begin{cases}2k(1+d)e^x=0 \\ k^2e^{2x}=-e^{2x} \\ d^2+d=0\end{cases}\Rightarrow\begin{matrix}k=\pm i \\ d=-1\end{matrix}\Rightarrow H(x)=\pm ie^x\end{cases}$$

$$y_h=e^{\int vdx}(A\cos\int u\,dx+B\sin\int u\,dx)=(A\cos\int e^xdx+B\sin\int e^xdx)=$$
$$(Acose^x+Bsine^x)$$

At the end, it should be noted that simplification of $R(x)$ to its primary elements and alternation of the variable may lead to elimination of HR system's solutions. So, in case of arriving at no answer for HR system, it is recommended to focus on reaching A, B, K, C, G, S, and D coefficients.

CONCLUSION

By provision of HR system, it was attempted to facilitate the process of guessing in arriving at one solution of the differential equation. Now, to what extend the process is facilitated or become harder depends on the user of HR system, and cannot be clearly judged. In the methods designed for solving HR system, resolving this ambiguity that in what situation solving HR system, by using the assumed $R(x)$, would result in, at least, one solution of the differential equation, is done by giving an example: if we set $R(x)=\frac{a}{x}+b+p(x)$ and constants a and b are found at the end of the process of solving the system, then Or b and a are attained, in turn, as real and complex constants, $a = p \pm iq$ (first condition), then

$$H(x) = R(x) - p(x) = \frac{a}{x} + b + p(x) - p(x) = \frac{p \pm iq}{x} + b = \frac{\pm iq}{x} + \frac{p}{x} + b$$
$$y_h = e^{\int \frac{p}{x} + b dx}(A \cos \int \frac{q}{x} dx + B \sin \int \frac{q}{x} dx) = x^p e^{bx}(A \cos qLnx + B \sin qLnx)$$

Or a and b are reached, in turn, as real and complex constants, $b = p \pm iq$ (second condition), then

$$H(x) = R(x) - p(x) = \frac{a}{x} + b + p(x) - p(x) = \frac{a}{x} + p \pm iq$$
$$y_h = e^{\int \frac{a}{x} + p dx}(A \cos \int q\, dx + B \sin \int q\, dx) = x^a e^{px}(A \cos qx + B \sin qx)$$

Or both a and b are attained as real constants (third condition), then

$$y = ce^{\int H(x)dx} = ce^{\int \frac{a}{x} + b dx} = ce^{\int \frac{a}{x} + b\, dx} = ce^{lnx^a + bx} = cx^a e^{bx}$$

It can be concluded that by setting $R(x) = \frac{a}{x} + b + p(x)$, the solution of the system would reach, at least, one answer of the differential equation. In this case, at least one of the solutions would be $f(n)_1$, $f(n)_2$, $f(n)_3$, or $f(n)_4$, which are in following forms

$$f(n)_1 = x^n$$
$$f(n)_2 = e^{nx}$$
$$f(n)_3 = A \cos nLnx + B \sin nLnx$$
$$f(n)_4 = A \cos nx + B \sin nx$$

Or would be a multiple of $f(n)_1$, $f(n)_2$, $f(n)_3$, $f(n)_4$, where this multiplication conforms to one the first, second, or third conditions. By setting $R(x) = \frac{a}{x} + b + p(x)$ and substituting it into the *HR* system, we have

$$HR \begin{cases} R(x) - H(x) = p(x) \Rightarrow H(x) = R(x) - p(x) = \frac{a}{x} + b \\ R'(x) + R(x)H(x) = p'(x) - Q(x) \Rightarrow \frac{-a}{x^2} + p'(x) + (\frac{a}{x} + b + p(x))(\frac{a}{x} + b) = p'(x) - Q(x) \\ \Rightarrow -\frac{a}{x^2} + (\frac{a}{x} + b + p(x))(\frac{a}{x} + b) = -Q(x) \end{cases}$$

Given the solution of the *HR* system with the assumption $R(x) = \frac{a}{x} + b + p(x)$, it can be said that if we guesse y in the first, second, or third conditions, and then substitute them into the differential Equation 1, so as to examine the guesses, we would have to perform time-consuming calculations. While, by summarizing these guesses, using the simple assumption $R(x) = \frac{a}{x} + b + p(x)$, we can examine the guesses using shorter calculations of *HR* system. Generally, if we guess y_1 as one solution of the homogeneous differential equation, and then compute the y_1-equivalent $R(x)$ and substitute it into the *HR* system so as to find out whether the guess is correct, then facilitation of the process of guess examination calculation depends on y_1.

We know that:

$$R(x) = Ap(x) + BQ(x) + \sqrt{KQ(x)} + C\frac{Q(x)}{p(x)} + G\frac{p(x)}{Q(x)} + \frac{S}{p(x)} + D$$

Now, by setting $K = G = S = 0$ and $\frac{Q(x)}{p(x)} = \frac{1}{x}$, we would have:

$$R(x) = Ap(x) + BQ(x) + \frac{C}{x} + D \quad (12)$$

Also, by setting $B = C = G = S = D = 0$ and $A = 1$

$$R(x) = p(x) + \sqrt{KQ(x)} \quad (13)$$

By using Equation 12 or 13, the corresponding *HR* system of all differential equations presented in the third chapter of "*The Differential Equations*", by Nikokar, 3rd ed., Azadeh Publication, are solvable.

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