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ORIGINAL ARTICLE

NUMERICAL SOLUTION OF RICCATI DIFFERENTIAL EQUATION USING LEGENDRE SCALING FUNCTIONS

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ABSTRACT: A numerical technique is presented for the solution of Riccati differential equation. This method uses the Legendre scaling functions. The method consists of expanding the required approximate solution as the elements of Legendre scaling functions. Using the operational matrix of integral, we reduce the problem to a set of algebraic equations. Some numerical example is included to demonstrate the validity and applicability of the technique. The method is easy to implement and produces very accurate results.

KEYWORDS: Riccati differential equation, collocation method, Legendre scaling functions, operational matrix of integral.

INTRODUCTION

In this paper, a numerical method using Legendre scaling functions is presented for the following Riccati differential equation:

$$u'(t) = p(t) + q(t)u(t) + r(t)u^2(t), \qquad 0 \le t \le 1, \tag{1.1}$$

With initial value

$$u(0) = \alpha, \tag{1.2}$$

Which, plays a significant role in many fields of applied science (Reid, 1972). For example, as is well-known, a one-dimensional static Schrodinger equation is closely related to a Riccati differential equation. Solitary wave solution of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation (Carinena *et al.,* 1998). Such types of problems also arise in the optimal control literature. Therefore, the problem has attracted much attention and has been studied by many authors. However, deriving its analytical solution in an explicit form seems to be unlikely except for certain special situations. For example, some Riccati equations with constant coefficients can be solved analytically by various methods (Scott, 1973). Therefore, one has to go for numerical techniques or approximate approaches for getting its solution. Recently, Adomian's decomposition method has been proposed for solving Riccati differential equations in (El-Tawil *et al.,* 2004). Abbasbandy, (2006a); Abbasbandy, (2006b) and Abbasbandy, (2007) solved a special Riccati differential equation-quadratic Riccati differential equation using He's variational iteration method, homotopy perturbation method and iterated He's homotopy perturbation method and compared the accuracy of the obtained solution with that derived by Adomian decomposition method. Geng *et al.,* (2009) introduce a piecewise variational iteration method for Riccati differential equations, which is a modified variational iteration method. Guo, (2003); Guo *et al.,* (2005); Guo and Zhang, (2007) and Guo and Yan, (2009) used spectral methods to solve differential equations numerically. Two numerical methods based on Cubic B-spline scaling functions and Chebyshev cardinal functions were introduced by Lakestani and Saray, (2012). Also Dehghan *et al.,* (2011) studied the global behavior of the Riccati difference equation of order two. A GMRES-based BDF method for solving differential Riccati equations presented by Hernández *et al.,* (2008). Peinado *et al.,* (2010) introduced Adams–Bashforth and Adams–Moulton methods for solving differential Riccati equations. A comparative study of numerical methods for solving quadratic Riccati differential equations presented by Mohammadi and Hosseini, (2011). Tsai and Chen, (2010) introduced an approximate analytic solution of the nonlinear Riccati differential equation.

In this work we reduce the problem to a set of algebraic equations by expanding the unknown function as Legendre scaling functions, with unknown coefficients. The operational matrix of integral is given. This matrix together the Legendre scaling functions are then utilized to evaluate the unknown coefficients.

This paper is organized as follows: In Section 2, we describe the formulation of the Legendre scaling functions on [0, 1], then derive the operational matrix of integral required for our subsequent development. In Section 3 the proposed method is used to approximate the solution of the problem in interval [0, 1]. As a result a set of algebraic equations are formed and a solution of the considered problem is introduced.

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In Section 4, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting numerical examples. Section 5, ends this paper with a brief conclusion. Note that we have computed the numerical results by Maple programming.

LEGENDRE SCALING FUNCTIONS

The r-th order Legendre scaling functions are the set of r+1 functions $\phi^0(x),\ldots,\ \phi^r(x)$ here $\varphi^i(x)$ is a polynomial of i-th order Xufeng and Danfu, (2007); Khellat and Yousefi, (2006); Singh *et al.*, (2008); Shamsi and Razzaghi, (2005) and Alpert *et al.*, (2002) and all ϕ's form orthonormal basis, that is, for i=0, 1, ...,r,

$$\phi^i(x) = \sum_{k=0}^{i} a_{ij}\, x^k, \qquad i = 0,1,\cdots,r \tag{2.1}$$

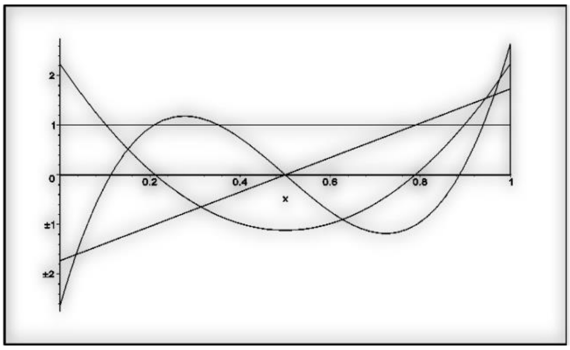
Where,

$$\int_0^1 \phi^i(x)\phi^k(x)dx \qquad i,k = 0,1,\cdots,r \tag{2.2}$$

In fact the variables $a_{\{ik\}}$ are chosen so that the functions $\phi^i(x)\,, i = 0,1,\cdots,r$ form an orthonormal basis. For example the cubic Legendre scaling functions (r=3), consist of four functions as:

$$\begin{cases} \phi_0(x) = 1, & 0 \le x < 1 \\ \phi_1(x) = \sqrt{3}(2x-1)\,, & 0 \le x < 1 \\ \phi_2(x) = \sqrt{5}(6x^2-6x+1), & 0 \le x < 1 \\ \phi_3(x) = \sqrt{7}(20x^3-30x^2+12x-1), & 0 \le x < 1 \end{cases} \tag{2.3}$$

Figure 1, shows the plot of cubic Legendre scaling functions.



Scaling function for r=3

2.1. Function Approximation

For a fixed J, a function f(t) defined over [0,1] can be approximated using Legendre scaling functions of order r as,

$$f(x) = \sum_{k=0}^{2^J-1}\sum_{m=0}^{r} c_{J,k}\phi_{J,k}^m(x) = C^T\Phi_J(x) \tag{2.4}$$

Where,

$$C = \left[\left\{c_{J0}^0,\ldots,c_{J0}^0\right|\ldots\left|c_{J,2^J-1}^0,\ldots,c_{J,2^J-1}^r\right\}\right]^T, \tag{2.5}$$

$$\Phi_J(x) = \left[\phi_{J0}^0(t),\ldots,\phi_{J0}^r(t)\right|\ldots\left|\phi_{J,2^J-1}^0(t),\ldots,\phi_{J,2^J-1}^r(t)\right]^T \tag{2.6}$$

With,

$$c_{Jl}^m = \int_0^1 f(t)\phi_{Jk}^m(x)dx\,,\ \ k = 0,1,\cdots,2^J-1,\ \ m = 0,1,\cdots,r, \tag{2.7}$$

And

$$\phi_{Jk}^i(x) = \phi^i(2^{Jx}-k), \quad k = 0,1,\cdots,2^J-1, \qquad i = 0,1,\cdots,r.$$

The dimensions of vectors C and $\phi^J(x)$ are $r2^J$.

2.2. The Operational Matrix of Integral

The integral of vector $\Phi_J(x)$ in (2.6) can be expressed as:

$$\int_0^x \Phi_J(x)dx = I_\phi\Phi_J(x), \tag{2.8}$$

Where, I_ϕ is $n\times n$ a operational matrix of integral for Legendre scaling functions. The matrix I_ϕ can be obtained by the following process. Let:

$$a_0 = 1$$

$$a_i = \sqrt{a_{i-1}^2+2} \qquad i = 1,2,\ldots$$

$$b_i = \frac{1}{a_{i-1}a_i} \qquad i = 1,2,\ldots$$

$$A_r = \begin{bmatrix} 0 & b_1 & 0 & & & & & \\ & 0 & b_2 & 0 & & & & \\ & & 0 & b_3 & 0 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 0 & b_{r-2} & 0 \\ & & & & & 0 & b_{r-1} \\ & & & & & & 0 \end{bmatrix}_{\{r\times r\}}$$

$$B_r = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{r\times r}$$

And

$$M_r = \frac{1}{2^{J+1}}(A_r - A_r^T + 2^J B_r)$$

Now the operational matrix of Integral I_ϕ can be calculated as:

$$I_\phi = \frac{1}{2^J}\begin{bmatrix} M_r & B_r & \cdots & B_r \\ & \ddots & \ddots & \vdots \\ & & M_r & B_r \\ & & & M_r \end{bmatrix}_{n\times n}$$

DESCRIPTION OF NUMERICAL METHOD

In this section, we solve Riccati differential equations of the form (1.1) with the initial value (1.2), by using Legendre scaling functions. For this purpose, we first assume

$$z(x) = u^2(x),\ 0 \le x \le 1. \tag{3.1}$$

We now use Eq. (2.4) to approximate z(x) as:

$$z(x) = Z^T\Phi_J(x), \tag{3.2}$$

Where, Z is a $(n \times 1)$ unknown vector difined similarly to C in Eq. (2.4).
Let:

$$u'(x) = U^T \Phi_J(x) \quad (3.3)$$

Where, U is a $(n \times 1)$ unknown vector and should be found. By integrating from both sides of Eq. (3.3) we get

$$u(x) - u(0) = U^T \int_0^x \Phi_J(t)dt = U^T I_\phi \Phi_J(x). \quad (3.4)$$

Using the initial value (1.2) in Eq. (3.4), we get

$$u(x) = U^T I_\phi \Phi(x) + \alpha \quad (3.5)$$

Also, the functions p(x), q(x) and r(x) in Eq. (1.1) using Eq. (2.4), can be approximated as

$$p(x) = P^T \Phi(x) \quad (3.6)$$

$$q(x) = Q^T \Phi(x) \quad (3.7)$$

And

$$r(x) = R^T \Phi(x) \quad (3.8)$$

Where, P, Q and R, are $(n \times 1)$ vectors defined as:

$$P = [p_{J0}^0, \dots, p_{J0}^0 | \dots \left| p_{J,2^J-1}^0, \dots, p_{J,2^J-1}^r \right]^T,$$
$$Q = [q_{J0}^0, \dots, q_{J0}^r | \dots \left| q_{J,2^J-1}^0, \dots, q_{J,2^J-1}^r \right]^T,$$

And

$$R = [r_{J0}^0, \dots, r_{J0}^r | \dots \left| r_{J,2^J-1}^0, \dots, r_{J,2^J-1}^r \right]^T,$$

The entries of the vectors P, Q and R can be found similar to Eq. (2.7). Applying, Eqs. (3.2)-(3.8) in Eq. (1.1), we get

$$U^T \Phi_J(x) - P^T \Phi_J(x) - Q^T \Phi_J(x) \Phi_J(x)^T I_\phi^T U - Q^T \Phi_J(x) \alpha - R^T \Phi_J(x) \Phi_J^T(x) Z = 0 \quad (3.9)$$

Suppose, $x_{i+1} = \frac{i}{n-1}$, $i = 0, \dots, n-1$, and

$$N_{i+1} = \Phi_J(x_{i+1}) \Phi_J^T(x_{i+1}), \quad i = 0, \dots, n-1 \quad (3.10)$$

The block matrices N_{i+1} can be obtained by the following process. Let $\widehat{\Delta}_{i+1} = \hat{\delta}_{j,k}^{i+1}$ be a $(r \times r)$ matrix defined as:

$$\hat{\delta}_{1,1}^{i+1} = 1,$$

$$\hat{\delta}_{1,2}^{i+1} = (2x_{i+1} - 1)\sqrt{3},$$

$$\hat{\delta}_{1,l}^{i+1} = \left(\frac{(2l-3)(2x_{i+1}-1)}{l-1} \frac{\hat{\delta}_{1,l-1}^{i+1}}{\sqrt{2l-3}} - \frac{(l-2)}{l-1} \frac{\hat{\delta}_{1,l-2}^{i+1}}{\sqrt{2l-5}} \right) \sqrt{2l-1},$$
$$l = 3, 4, \cdots, r,$$

And

$$\hat{\delta}_{\{m,\tilde{m}\}}^{i+1} = \hat{\delta}_{\{1,m\}}^{i+1} \; \hat{\delta}_{\{1,\tilde{m}\}}^{i+1} \qquad m = 2, \dots, r, \qquad \tilde{m} = 1, \dots, r,$$

Now suppose

$$\Delta_{\{3(\hat{\imath}-1)+\hat{\jmath}\}} = \widehat{\Delta}_{\{4(\hat{\jmath}-1)+\hat{\imath}\}} \quad \hat{\imath} = 1, \dots, 2^J, \qquad \hat{\jmath} = 1, \dots, r.$$

The matrices N_{i+1} , $i = 0, 1, \cdots, n-1$ in Eq. (3.10) can be expressed:

$$\begin{cases} N_{\{i+1\}} = 2^J . \operatorname{diag}[\Delta_{i+1}, 0_{\{r,r\}}, \dots, 0_{\{r,r\}}], & i = 0, \cdots, r-1, \\ N_{\{i+1\}} = 2^J . \operatorname{diag}[0_{\{r,r\}}, \Delta_{i+1}, 0_{\{r,r\}}, \dots, 0_{\{r,r\}}], & i = r, \cdots, 2r-1, \\ \vdots \\ N_{\{i+1\}} = 2^J . \operatorname{diag}[0_{\{r,r\}}, \dots, 0_{\{r,r\}}, \Delta_{i+1}], & i = r2^{\{J-1\}}, \dots, 2^J r - 1 \end{cases} \quad (3.11)$$

Where, $O_{\{r,r\}}$ is $r \times r$ zero matrix.
Let, $\theta_{i+1} = \Phi_J(x_{i+1}), i = 0, 1, \cdots, n-1$, be $(n \times 1)$ vectors. The entries of vectors $\hat{\theta}_{i+1}$ can be found in the following way.
Suppose, $\hat{\theta}_{i+1} = (\hat{\theta}_{\{j,k\}})$, $i = 0, \dots, n-1$ be $(1 \times r)$ vectors with the following entries:

$$\hat{\theta}\{_{\{1,l\}}^{i+1} = \delta_{\{1,l\}}^{i+1},$$

Now it can be shown that

$$\begin{cases} \theta_{\{i+1\}} = [\hat{\theta}_{i+1}, 0_{\{r\}}, \dots, 0_{\{r\}}]^T, & i = 0, \dots, r-1, \\ \theta_{\{i+1\}} = [0_{\{r\}}, \hat{\theta}_{i+1}, 0_{\{r\}}, \dots, 0_{\{r,r\}}]^T & , i = r, \dots, 2r-1 \\ \vdots \\ \theta_{\{i+1\}} = [0_{\{r\}}, \dots, 0_{\{r\}}, \hat{\theta}_{i+1}]^T, & i = r2^{\{J-1\}}, \dots, 2^J r - 1, \end{cases} \quad (3.12)$$

Where, O_r is $(1 \times r)$ zero vector.
By collocating Eq. (3.9) in *points* $x_{i+1}, i = 0, \cdots, n-1$, and using Eqs. (3.11) and (3.12) we get

$$U^T \theta_{i+1} - P^T \theta_{i+1} - Q^T N_{i+1} I_\phi^T U - Q^T \theta_{i+1} \alpha - R^T N_{i+1} Z = 0. \quad (3.13)$$

Putting Eq. (3.5) in Eq. (3.1) and using Eq. (3.2) we have

$$\left(U^T I_\phi \Phi_J(x) + \alpha \right)^2 = Z^T \Phi_J(x). \quad (3.14)$$

Collocating Eq. (3.14) in points $x_i, i = 0, 1, \cdots, n-1$ we get

$$\left(U^T I_\phi \theta_{i+1} + \alpha \right)^2 = Z^T \theta_{i+1}. \quad (3.15)$$

Eq. (3.14) together with Eq. (3.15) give a system of nonlinear equations with 2n equations and unknowns, which can be solved to find the entries of vectors U and Z. So the unknown function u(x) can be found using Eq. (3.5).

NUMERICAL EXAMPLE

In this section we give some computational results of numerical experiments with methods based on preceding section, to support our theoretical discussion. The nonlinear systems obtained by collocation method are solved by newton method.

Example 1: Consider the following Riccati differential equation:

$$\begin{cases} u'(x) = 1 + x^2 - u^2(x), & 0 \leq x \leq 1, \\ u(0) = 1. \end{cases}$$

The exact solution is given in Geng *et al.*, (2009) as:

$$u(x) = x + \frac{x^{\{-x^2\}}}{1 + \int_0^1 e^{\{-t^2\}} dt}.$$

Table 1 shows the absolute values of errors for different values of r and J, using the method presented in the previous section.

Table 1: Absolute values of errors

x	r=2 J=3	r=3 J=3	r=4 J=3
0.0	3.0×10^{-5}	5.1×10^{-7}	7.0×10^{-9}
0.1	1.0×10^{-5}	2.1×10^{-7}	1.1×10^{-9}
0.2	6.5×10^{-6}	9.2×10^{-8}	1.4×10^{-9}
0.3	4.6×10^{-6}	8.0×10^{-8}	1.0×10^{-9}
0.4	4.7×10^{-6}	1.2×10^{-7}	1.2×10^{-9}
0.5	1.7×10^{-6}	2.7×10^{-8}	6.6×10^{-10}
0.6	3.0×10^{-6}	1.2×10^{-7}	1.9×10^{-9}
0.7	1.4×10^{-6}	6.4×10^{-8}	1.7×10^{-9}
0.8	2.2×10^{-8}	6.3×10^{-8}	3.3×10^{-9}
0.9	9.3×10^{-7}	1.0×10^{-7}	7.8×10^{-9}
1.0	2.9×10^{-6}	2.5×10^{-7}	1.7×10^{-8}

Example 2: Consider the following Riccati differential equation:

$$\begin{cases} u'(x) = 1 + 2u(x) - u^2(x), & 0 \leq x \leq 1, \\ u(0) = 0. \end{cases}$$

The analytical solution is given in El-Tawil *et al.*, (2004); Abbasbandy, (2006a); Abbasbandy, (2006b); Abbasbandy, (2007) and Geng *et al.*, (2009) as:

$$u(x) = 1 + \sqrt{2} tanh(\sqrt{2}x + \frac{ln(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2}).$$

Table 2 shows the absolute values of errors for different values of r and J, using the method presented in the previous section.

Table 2: Absolute values of errors

x	r=2 J=3	r=3 J=3	r=4 J=3
0.0	2.2×10^{-5}	1.7×10^{-6}	5.5×10^{-8}
0.1	7.3×10^{-6}	6.8×10^{-7}	1.2×10^{-8}
0.2	6.6×10^{-7}	6.7×10^{-7}	2.2×10^{-8}
0.3	1.7×10^{-5}	7.7×10^{-7}	8.7×10^{-9}
0.4	3.4×10^{-5}	1.1×10^{-6}	3.9×10^{-8}
0.5	2.9×10^{-5}	1.7×10^{-7}	4.6×10^{-8}
0.6	4.6×10^{-5}	4.7×10^{-7}	2.0×10^{-8}
0.7	3.6×10^{-5}	3.4×10^{-7}	1.0×10^{-8}
0.8	2.5×10^{-5}	6.3×10^{-7}	5.2×10^{-8}
0.9	2.0×10^{-5}	1.4×10^{-6}	6.5×10^{-8}
1.0	5.1×10^{-5}	3.3×10^{-6}	1.8×10^{-7}

Example 3: Consider the following Riccati differential equation:

$$\begin{cases} u'(x) = \frac{-1}{1+x} + u(x) - u^2(x), & 0 \leq x \leq 1, \\ u(0) = 1. \end{cases}$$

The analytical solution is given in Lakestani and Saray, (2012) as:

$$u(x) = \frac{1}{1+x}.$$

Table 3 shows the absolute values of errors for r=3, J=3, using the method presented in the previous section and the method used in Lakestani and Saray, (2012).

Table 3: Absolute values of errors

x	Presented method	method used in [14]
0.1	1.0×10^{-6}	1.5×10^{-5}
0.2	8.7×10^{-8}	5.9×10^{-6}
0.3	3.5×10^{-7}	7.7×10^{-6}
0.4	8.3×10^{-7}	7.7×10^{-6}
0.5	6.2×10^{-7}	8.3×10^{-6}
0.6	7.3×10^{-7}	6.3×10^{-6}
0.7	5.3×10^{-7}	5.7×10^{-6}
0.8	5.5×10^{-7}	5.9×10^{-6}
0.9	6.5×10^{-7}	5.9×10^{-6}
1.0	4.8×10^{-7}	5.7×10^{-6}

Example 4: Consider the following Riccati differential equation:

$$\begin{cases} u'(x) = u(x) - 2u^2(x), & 0 \leq x \leq 1, \\ u(0) = 1. \end{cases}$$

The analytical solution is given in Lakestani and Saray, (2012) as:

$$u(x) = \frac{1}{2 - e^{-x}}.$$

Table 4 shows the absolute values of errors for r=4, J=3, using the method presented in the previous section and the method used in Lakestani and Dehghan, (2010).

Table 4: Absolute values of errors

x	Presented method	Method used In [16]	
		N=5	N=6
0.1	4.8×10^{-7}	4.8×10^{-7}	4.8×10^{-7}
0.2	4.8×10^{-7}	4.8×10^{-7}	4.8×10^{-7}
0.3	4.8×10^{-7}	4.8×10^{-7}	4.8×10^{-7}
0.4	4.8×10^{-7}	4.8×10^{-7}	4.8×10^{-7}
0.5	4.8×10^{-7}	4.8×10^{-7}	4.8×10^{-7}

CONCLUSION

In this paper we presented a numerical scheme for solving the Riccati differential equation. The Legendre scaling functions was employed. The obtained results showed that this approach can solve the problem effectively.

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