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ORIGINAL ARTICLE

Some Properties of GC-Fusion Frames

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ABSTRACT. In this article we will discuss about gc-fusion frame bounds. We present a new method to obtain gc-fusion frame bounds. In the applications to irregular sampling of band-limited functions this alternative strategy leads to better and explicit estimates of the gc-fusion frame bounds.

Keywords: Operator, Bounded Operator, Iterative Reconstruction, fusion frames, gc-fusion frames Operator. 2000 Mathematics Subject Classification. Primary: 42C15; Secondary 46L30.

Introduction

Frames are generalizations of bases which lead to redundant signal expansions. A frame for a Hilbert spaces a set of not necessarily linearly independent vectors that spans and has some additional properties. Frames were first introduced in (Duffin et al., 1952) in the context of nonharmonic Fourier series, and play an important role in the theory Fourier series, and play an important role in the theory of nonuniform sampling.

The notion fusion frames, introduced by authors in (Casazza and Kutyniok, 2008) provides an extensive framework not only to model sensor networks, but also to provide a means to improve robustness or develop feasible reconstruction algorithms. Related approaches with a different focus were undertaken by (Aldroubi et al., 2004; Fornasier, 2004).

The organization of this article is as follows. In section 2, we express the definitions frame and fusion for a Hilbert space with their most important properties. In section 3, with name main result, first we will review gc-fusion frames from (Faroughi, 2010). We will present a new method to obtain gc-fusion frame bounds and in the applications to irregular sampling of band-limited functions this alternative strategy leads to better and explicit estimates the gc-fusion frames.

Preliminaries

Definition

Let $\{f_i\}_{i\in I}$ be sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H if there exist $0<A\le B<\infty$ such that for all $h\in H$,

$$A\|h\|^2\le\sum_{i\in I}|<f_i,h>|^2\le B\|h\|^2.$$

The constants A and B are called frame bounds. If A,B can be chosen so that $A=B$, we call this frame an $A-tight$ frame and if $A=B=1$ it is called a parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence.

If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded,

$T:l^2(I)\to H, T(c_i)=\sum_{i\in I}c_if_i$ (synthesis operator),

$T^*:H\to l^2(I), T^*f=\{<f,f_i>\}_{i\in I}$ (analysis operator),

$S:H\to H, Sf=TT^*f=\sum_{i\in I}<f,f_i>f_i$ (frame operator).

Definition

For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspace in H, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i>0$ for all $i\in I$. Then $\{(W_i,v_i)\}_{i\in I}$ is a frame of subspaces for H if there exist $0<C\le D<\infty$ such that:

$$C\|f\|^2\le\sum_{i\in I}v_i^{\,2}\|\pi_{W_i}(f)\|^2\le D\|f\|^2, h\in H,$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. We call C and D the frame of subspaces bounds. The family $\{(W_i,v_i)\}_{i\in I}$ is called a $C-tight$ frame of subspaces, if in above inequality the constants C and D can be chosen so that $C=D$, a parseval frame of subspaces provided $C=D=1$ and an orthonormal frame of subspaces basis if $H=\oplus_{i\in I}W_i$. If $\{(W_i,v_i)\}_{i\in I}$ possesses an upper frame of subspaces bound, but not necessarily a lower bound, we call it a Bessel frame of subspaces sequence with Bessel frame of subspaces bound D. Also a family of subspaces $\{W_i\}_{i\in I}$ of H is called complete, if $H=\overline{span}\{W_i\}$. The representation space employed in this setting is

$$(\sum_{i\in I}\oplus W_i)_{l^2}=\{\{f_i\}\,|\,f_i\in W_i \quad and \quad \{f_i\}\in l^2(I)\}.$$

Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. The operators are employed, which are defined by

$$T:(\sum_{i\in I}\oplus W_i)_{l^2}\to H,\, T(\{f_i\})=\sum_{i\in I}v_i f_i$$

(synthesis operator),

$$T^*:H\to(\sum_{i\in I}\oplus W_i)_{l^2},\, T^*(h)=\{v_i\pi_{W_i}(h)\}_{i\in I}$$

(analysis operator),

$$S:H\to H,\, S(h)=TT^*(h)=\sum_{i\in I}v_i^2\pi_{W_i}(h)$$

(frame operator).

Main Result

Definition

Let $F:X\to\hat{H}$ be such that for each $h\in H$, the mapping $x\to\pi_{F(x)}(h)$ is measurable (i.e. F is weakly measurable) and let $\{K_x\}_{x\in X}$ be a collection of Hilbert spaces. For each $x\in X$, suppose that $\Lambda_x\in B(F(x),K_x)$ and put

$$\Lambda=\{\Lambda_x\in B(F(x),K_x):x\in X\}.$$

Then (Λ,F,v) is a gc-fusion frame for H if there exist $0<A\le B<\infty$ such that

$$A\|h\|^2\le\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu\le B\|h\|^2$$

for all $h\in H$, where $\pi_{F(x)}(h)$ is the orthogonal projection onto the subspace $F(x)$.

(Λ,F,v) is called a tight gc-fusion frame for H if A,B can be chosen so that $A=B$, and parseval if $A=B=1$. If just the right hand inequality satisfies then we say that (Λ,F,v) is a Bessel gc-fusion mapping for H.

Let $K=\oplus_{x\in X}K_x$ and $L^2(X,K)$ be a collection of all measurable functions $\varphi:X\to K$ such that for each $x\in X$, $\varphi(x)\in K_x$ and

$$\int_X\|\varphi(x)\|^2 d\mu<\infty.$$

It can be verified that $L^2(X,K)$ is a Hilbert space with inner product defined by

$$<\varphi,\gamma>=\int_X\langle\varphi(x),\gamma(x)\rangle d\mu$$

for $\varphi,\gamma\in L^2(X,K)$ and the representation space in this setting is $L^2(X,K)$.

Remark

Let (Λ,F,v) be aBessel gc-fusion mapping with Bessel bound B, $\varphi\in L^2(X,K)$ and $h\in H$. Then

$$\left|\int_X v(x)\,\langle\Lambda_x^*(\varphi(x)),h\rangle d\mu\right|=$$

$$\left|\int_X v(x)\,\langle\Lambda_x^*(\varphi(x)),\pi_{F(x)}(h)\rangle d\mu\right|$$

$$=\left|\int_X v(x)\,\langle\varphi(x),\Lambda_x(\pi_{F(x)}(h))\rangle d\mu\right|$$

$$=\left|\int_X v(x)\,\|\varphi(x)\|.\|\Lambda_x(\pi_{F(x)}(h))\|d\mu\right|$$

$$\le(\int_X\|\varphi(\mathrm{x})\|^2 d\mu)^{1/2}).(\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu)^{1/2}$$

$$\le B^{1/2}\|h\|(\int_X\|\varphi(\mathrm{x})\|^2 d\mu)^{1/2}).$$

So we may define:

Definition

Let (Λ,F,v) be aBessel gc-fusion mapping for H. We define the gc-fusion pre-frame operator (synthesis operator) $T_{gf}:L^2(X,K)\to\mathrm{H}$, by

$$\langle T_{gf}(\varphi),h\rangle=\int_X v(x)\,\langle\Lambda_x^*(\varphi(x)),h\rangle d\mu,$$

Where $\varphi\in L^2(X,K)$ and $h\in H$. It is obvious that T_{gf} is linear and by remark 3.3 T_{gf} is a bounded linear mapping. Its adjoint $T_{gf}^*:\mathrm{H}\to L^2(X,K)$, will called gc-fusion analysis operator, and $S_{gf}=T_{gf}\circ T_{gf}^*$ will be called gc-fusion operator. For each $\varphi\in L^2(X,K)$ and $h\in H$, we have

$$\langle T_{gf}^*(h),\varphi\rangle=\langle \mathrm{h},T_{gf}(\varphi)\rangle=\int_X v(x)\,\langle h,\Lambda_x^*(\varphi(x))\rangle d\mu$$

$$=\int_X v(x)\,\langle\pi_{F(x)}(h),\Lambda_x^*(\varphi(x))\rangle d\mu=\int_X v(x)\,\langle\Lambda_x(\pi_{F(x)}(h)),\varphi(x)\rangle d\mu$$

$$=\langle v\Lambda_{(.)}\pi_{F(.)}(h),\varphi\rangle.$$

Hence for each $h\in H$,

$$T_{gf}^*=v\Lambda_{(.)}\pi_{F(.)}.$$

Definition

For each Bessel gc-fusion mapping (Λ,F,v) for H, we shall denote

$$A_{\Lambda,v}=\inf_{h\in H_1}\|v\Lambda_{(.)}\pi_{F(.)}(h)\|^2,$$

$$B_{\Lambda,v} = \sup_{h\in H_1} \|v\,\Lambda_{(.)}\pi_{F(.)}(h)\|^2 = \|v\,\Lambda_{(.)}\pi_{F(.)}\|^2 .$$

Remark

Let (Λ, F, v) is a Bessel gc-fusion mapping for H. Since, for each $h \in H$

$\langle T_{gf}T_{gf}^*(h),\varphi\rangle = \|v\,\Lambda_{(.)}\pi_{F(.)}(h)\|^2$, ,

$A_{F,v}$ and $B_{F,v}$ are optimal scalars which satisfy

$A_{F,v} \le T_F T_F^*(h) \le B_{F,v}$.

So (Λ, F, v) is a gc-fusion frame for H if and only if $A_{F,v} > 0$.

From the functional analytic point of view they are just their version of a linear operator by a Neumann series. A particular case of these methods are the algorithms based on frames. In [16], a new method is obtained for constructing frames which will be very useful obtain explicit estimates. In this article we extended it to gc-fusion frames. First we review the following proposition from [18] with its proof.

Proposition

Let A be a bounded operator on a Banach space $(B, \|.\|_B)$ that satisfies for some positive constant $\gamma < 1$

$$\|f - Af\|_B \le \gamma\|f\| \text{ for all } f \in B. \qquad (1)$$

Then A is invertible on B and f can be recovered from Af by the following iteration algorithm. Setting $f_0 = Af$ and

$$f_{n+1} = f_n + A(f - f_n) \qquad (2)$$

For $n \ge 0$, we have

$$\lim_{n\to\infty} f_n = f \qquad (3)$$

With the error estimate after n iterations

$$\|f - f_n\|_B \le \gamma^{n+1}\|f\|_B . \qquad (4)$$

Proof. By inequality (1) the operator norm of $Id - A$ is less than γ. This implies that A is invertible and that the inverse can be presented as a Neumann series:

$$A^{-1} = \sum_{n=0}^{\infty}(Id - A)^n$$

and any $f \in B$ is determined by Af and the norm-convergent series

$$f = A^{-1}Af = \sum_{n=0}^{\infty}(Id - A)^n Af .$$

The reconstruction (3) and the error estimate (4) follow easily after we have shown that the $n-th$ approximation f_n as defined in (2) coincides with the $n-th$ partial sum $\sum_{k=0}^{n}(Id - A)^k Af$. This is clear for $n = 0$, since $f_0 = Af$ by definition. Next assume that we know already that

$$f = \sum_{k=0}^{n}(Id - A)^k Af .$$

Then we obtain for $n+1$

$$\sum_{k=0}^{n+1}(Id - A)^k Af = Af + \sum_{k=1}^{n+1}(Id - A)^k Af$$

$$= Af + (Id - A)\sum_{k=0}^{n}(Id - A)^k Af \qquad (by \quad induction)$$

$$= Af + (Id - A)f_n = f_n + A(f - f_n).$$

Now clearly $\lim_{n\to\infty} f_n = f$ and since $\sum_{k=n+1}^{\infty}(Id - A)^k = (Id - A)^{n+1}A^{-1}$, we obtain

$$\|f - f_n\|_B = \left\|\sum_{k=n+1}^{\infty}(Id - A)^k Af\right\|_B = \left\|(Id - A)^{n+1}A^{-1}Af\right\| \le \gamma$$

Definition

Let (Λ, F, v) be a gc-fusion frame for H with bounds A, B and gc-fusion frame operator S_{gf} .We define quasi gc-fusion frame operator Γ_{gf} for gc-fusion frame (Λ, F, v) as follows

$$\Gamma_F(h) = \frac{2}{A+B}\int_X v(x)\Lambda_x^*(v(x)\Lambda_x(\pi_{F(x)}(h)))d\mu.$$

It is clear that Γ_{gf} is positive. Now, let Id be the identity operator on H and we consider the self-adjoint operator $Id - \Gamma_{gf}$ on H.

Lemma

Let Γ_{gf} be the quasi gc-fusion frame operator for (Λ, F, v), then the operator norm of $Id - \Gamma_{gf}$ is smaller than 1 and we have

$$-\frac{B-A}{B+A}\|h\|^2 \le \langle (Id - \Gamma_{gf})(h), h\rangle \le \frac{B-A}{B+A}\|h\|^2 .$$

Proof. Directly calculate.

Since $Id - \Gamma_F$ is self-adjoint, (5) entails the operator norm of $Id - \Gamma_F$ on H is smaller than $\frac{B-A}{B+A} < 1.$

For S_{gf}^{-1}, we may write the reconstructions of h as follow

$$h = S_{gf}^{-1} S_{gf}(h) = \int_X v(x) S_{gf}^{-1}(\Lambda_x^*(v(x)\Lambda_x(\pi_{F(x)}(h))))d\mu$$

$$h = S_{gf} S_{gf}^{-1}(h) = \int_X v(x)\Lambda_x^*(v(x)\Lambda_x(\pi_{F(x)}(S_{gf}^{-1}(h))))d\mu$$

Note that the two frame bounds A and B play a vital role in the algorithm and determine the speed of convergence of the algorithm. It is therefore of practical importance to estimate A and B as sharp as possible. One is content with the mere existence of frame bounds A and B considers the family the family of operators that we define as follows.

Definition

Let (Λ, F, v) be a gc-fusion frame for H with bounds A, B and gc-fusion frame operator S_{gf}. We define $\lambda-$quasi frame operator $\Gamma_{\lambda gf}$ for (Λ, F, v) as follows

$$\Gamma_{\lambda F}(h) = \lambda \int_X v(x)\Lambda_x^*(v(x)\Lambda_x(\pi_{F(x)}(h)))d\mu,$$

where λ is the so-called relaxation parameter.

Remark

With an estimate similar to Lemma 3.8 one obtains

$$\|h - \Gamma_{\lambda F}\| \le \gamma(\lambda)\|h\|,$$

where $\gamma(\lambda) = \max\{1-\lambda A, \quad 1-\lambda B\}$ and $\gamma(\lambda) < 1$ for small values of λ. Here we present a new method to obtain frame bounds. In application to irregular sampling of band-limited functions this alternative strategy leads to better and explicit estimates of the gc-fusion frame bounds.

Definition

Let $F : X \to \hat{H}$ and $G : X \to \hat{H}$ are such that for each $h \in H$, the mapping $x \to \pi_{F(x)}(h)$ and $x \to \pi_{G(x)}(h)$ are measurable (i.e. F and G are weakly measurable) and let $\{K_x\}_{x\in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in B(F(x), K_x)$, $\Psi_x \in B(G(x), K_x)$ and put

$$\Lambda = \{\Lambda_x \in B(F(x), K_x) : x \in X\},$$

$$\Psi = \{\Psi_x \in B(G(x), K_x) : x \in X\}.$$

We define an approximation operator $A_{FG} : H \to H$ with respect to (Λ, F, v) and (Ψ, G, v) as follows

$$A_{FG}(h) = \int_X v(x)\Psi_x^*(v(x)\Lambda_x(\pi_{F(x)}(h)))d\mu.$$

Theorem

Suppose that F and G be defined as in Definition 3.6, for each $h \in H$ and $\varphi \in L^2(X, K)$ we have

$$\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu \le C_1\|h\|^2 \qquad (8),$$

$$\left\|\int_X v(x)\Psi_x^*(\varphi(x))d\mu\right\|^2 \le C_2\|\varphi\|_2^2 \qquad (9),$$

$$\left\|h - \int_X v(x)\Psi_x^*(v(x)\Lambda_x(\pi_{F(x)}(h)))d\mu\right\| \le \gamma\|h\|. \qquad (10)$$

Then F is a continuous frame with bounds $(1-\gamma)^2 / C_2$ and C_1, also G is a continuous frame with bounds $(1-\gamma)^2 / C_1$ and C_2.

Proof. Let A_{FG} be defined as in Definition 3.6, then A_{FG} is bounded operator on H because for each $h \in H$, assuming $\varphi(x) = v(x)\Lambda_x(\pi_{F(x)}(.))$ then (8) results in $\varphi \in L^2(X, \mu)$, and by (8) and (9) we have

$$\|A_{FG}(h)\|^2 = \left\|\int_X v(x)\Psi_x^*(v(x)\Lambda_x(\pi_{F(x)}(h)))d\mu\right\|^2$$

$$\le C_2\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu \le C_1C_2\|h\|^2.$$

By proposition 2.1 A_{FG} is invertible with $A_{FG}^{-1} = \sum_{n=0}^{\infty}(Id - A_{FG})^n$ and $\|A_{FG}^{-1}\| \le (1-\gamma)^{-1}$.

Now by (8) and (9) we have

$$\|h\|^2 = \|A_{FG}^{-1}A_{FG}(h)\|^2 \le (1-\gamma)^{-2}\|A_{FG}(h)\|^2$$

$$= (1-\gamma)^{-2}\left\|\int_X v(x)\Psi_x^*(v(x)\Lambda_x(\pi_{F(x)}(h)))d\mu\right\|$$

$$\le C_2(1-\gamma)^{-2}\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu \le C_1C_2(1-\gamma)^{-2}\|h\|^2.$$

We conclude that F has required properties.

Next we verify two inequalities which are dual to (8) and (9),

$$\left(\int_X v^2(x)\|\Psi_x(\pi_{G(x)}(h))\|^2 d\mu\right)^2 = \left(\int_X v^2(x)\left\langle\Psi_x(\pi_{G(x)}(h)), \Psi_x(\pi_{G(x)}(h))\right\rangle d\mu\right)^2$$

$$= \left(\left\langle\int_X v(x)\Psi_x^*(\Psi_x(v(x)\pi_{G(x)}(h)))d\mu, h\right\rangle\right)^2 \le \left\|\int_X v(x)\Psi_x^*(\Psi_x(v(x)\pi_{G(x)}(h)))d\mu\right\|^2\|h\|^2$$

$$\le C_2\left\|\int_X v^2(x)\Psi_x(\pi_{G(x)}(h))d\mu\right\|^2\|h\|^2$$

hence

$$\int_X v^2(x)\|\Psi_x(\pi_{G(x)}(h))\|^2 d\mu \le C_2\|h\|^2 .$$

For second inequality we have

$$\left\|\int_X v(x)\Lambda_x^*(\varphi(x))d\mu\right\| = \sup_{\|h\|=1}\left|\left\langle h, \int_X v(x)\Lambda_x^*(\varphi(x))d\mu\right\rangle\right|,$$

and

$$\left|\left\langle h, \int_X v(x)\Lambda_x^*(\varphi(x))d\mu\right\rangle\right|^2 = \left|\int_X v(x)\left\langle \Lambda_x(\pi_{F(x)}(h)), \varphi(x)\right\rangle d\mu\right|^2$$

$$\le \|\varphi\|_2^2 \int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 \le C_1\|\varphi\|_2^2 \|h\|^2 .$$

Now by similar argument and applying an approximation operator of the form

$$A_{GF}(h) = \int_X v(x)\Lambda_x^*(v(x)\Psi_x(\pi_{F(x)}(h)))d\mu.$$

Now, by similar method we can establish G has required properties.

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