JOURNAL OF CURRENT RESEARCH IN SCIENCE
ISSN 2322-5009
CODEN (USA): JCRSDJ
Available at www.jcrs010.com

JCRS
4 (3), 2016: 175-180

Existence Results For Non-Linear Fractiona Existence Results For Non-Linear Fractional Integro-Differential Equation With Non-Local Boundary Conditions

Asghar Ahmadkhanlu[1]*, Ahmad Pashavand[2]

1. Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
2. Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan.
Corresponding author email: ahmadkhanlu@azaruniv.edu

K E Y W O R D S: Fractional Derivatives, Caputo Differential Operator, Differential Equations, Boundary Value Problems, 2010 Mathematics Subject Classification: Primary 34AXX; Secondary 34A08

ABSTRACT: In this paper, we shall establish sufficient and necessary conditions for the existence of solutions for a first order boundary value problem for fractional differential equations including integral term in right hand side of the equation. This will be accomplished by Banach and Kransnoselskii fixed-point theorems.

Introduction

In recent years a considerable interest has been shown in the so-called fractional calculus, which allows us to consider integration and differentiation of any order, not necessarily integer [18]. In fact the fractional calculus can be considered an old and yet novel topic. Starting from some speculations of Leibniz and Euler, followed by the works of other eminent mathematicians including Laplace, Fourier, Abel, Liouville and Riemann, it has undergone a rapid development especially during the past two decades. (See [15,5]). Some results for fractional differential inclusions can be found in the book by Plotnikov et al [20]. For most details, we refer to the books by Podlubny [21] and Kilbass [17].

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [5]. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [4,7,9,13,17]).

On the other hand some generalization of fractional order differential equations have been done on time scales by authors [2] and also the well-posed of BVP and IVP for fractional order differential equations has been discussed with respectto the number of boundary conditions [14].

One of the emerging branches of this study is the theory of fractional quasi-linear equations, i.e. quasi-linear equations where the integer derivative with respect to time is replaced by a derivative of fractional order.

In this paper we consider the fractional boundary value problem including a quasi-linear equation with integral term in right hand side

$$ {}^{c}D^{\alpha}y(t) = f(t, y(s)) + \int_0^t g\,(t, s, y(s))ds, \qquad t \in [0, T] \tag{1}$$

$$ ay(0) + by(T) = c \tag{2}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, and a, b, c are real constants. Some existence results were given for the problem (1)-(2) with $g = 0$ by Benchohra and et all in [5] and initial value problem with Riemann-Liouville fractional operator by Furati and et all in [10].

Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. At first, we use the notation of $C(J, R)$ as a Banach space of continuous functions with the norm
$\| y \|_{\infty} = \sup\{\, |y(t)|, t \in J = [0, T]\}$.

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f: R^{+} \to R$ is defined by (the Abel-integral operator)

$$ I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}f(s)ds. \tag{3}$$

Provided that the integral exist.

Definition 2. The fractional derivative (in the sense of Caputo) of order $0 < \alpha < 1$ of a continuous function $f: R^{+} \to R$ is defined as the left inverse of the fractional integral of f

$$^{c}D^{\alpha}f(t) = I^{1-\alpha}\frac{d}{dt}f(t) \quad (4)$$

That is

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)ds, \quad (5)$$

provided that the right side exists.

For the theory and applications of fractional integrals and fractional derivatives we refer the reader to [17,21].

Theorem 3. (Kransnoselskii) Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $\mathcal{S}$. Suppose that A and B map $\mathcal{M}$ to $\mathcal{S}$ and that

$Ax + By \in \mathcal{M}$ $(\forall x, y \in \mathcal{M})$

A is a contraction mapping

B is a compact and continuous.

Then there exists $z \in \mathcal{M}$ such that $Az + Bz = z$

Proof. See [3]. □

Existence Results

Let us start by defining what we mean by solution of the problem (1)-(2).

Definition 4. A function $y \in C^1([0,T],\mathbb{R})$ is said to be a solution of (1)-(2) if y satisfies the equation

$$^{c}D^{\alpha}y(t) = f(t,y(t)) + \int_0^t g\,(t,s,y(s))ds$$

on J, and the condition

$$ay(0) + by(T) = c. \quad (6)$$

For the existence of solutions for the problem (1)-(2), we need the following auxiliary lemma:

Lemma 5. Let $0 < \alpha < 1$ and let $h: [0,T] \to R$ be continuous. A function y has a fractional integral form:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \quad (7)$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$$^{c}D^{\alpha}y(t) = h(t), \quad t \in [0,T], \quad (8)$$

$$y(0) = y_0. \quad (9)$$

Proof. See [5]. □

Lemma 6. A function $y \in C(J,\mathbb{R})$ is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s,y(s)))ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(f(s,y(s)))ds - c\right] \quad (10)$$

if and only if y is a solution of the following fractional BVP

$$\begin{cases} ^{c}D^{\alpha}y(t) = f(t,y) & 0 < \alpha < 1, \quad t \in J = [0,T] \\ ay(0) + by(T) = c \end{cases} \quad (11)$$

Proof. See [6]. □

As a consequence of Lemma 6 we have the following result which is useful in what follows.

Lemma 7. Let $0 < \alpha < 1$ and let $f: [0,T] \times R \to R$ and $g: [0,T] \times R \times R \to R$ be continuous functions. A function y is a solution of the fractional integral equation

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s)) + \int_0^s g\,(s,\tau,y(\tau))d\tau\right)ds - \frac{1}{a+b}\left[b\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s)) + \int_0^s g\,(s,\tau,y(\tau))d\tau\right)ds - c\right] \quad (12)$$

if and only if y is a solution of the fractional boundary value problem (1)-(2).

Our first result is based on Banach fixed point theorem.

Theorem 8. Assume that

- There exists a constant $k > 0$ such that

$|f(t,x) - f(t,y)| \le k|x-y|$, for each $t \in J$, and for all $x, y \in \mathbb{R}$

There exists $k' > 0$, $0 < \beta < 1$, $h: J \times \mathbb{R} \to \mathbb{R}$ such that

$$|g(t,s,x)-g(t,s,y)|\le (t-s)^{\beta-1}|h(s,x)-h(s,y)|\le k'(t-s)^{\beta-1}|x-y|$$

If

$$\frac{(k\beta+k'T^{\beta})|a+b|+(k+k')|b|T^{\beta}}{\alpha\beta\Gamma(\alpha)|a+b|}T^{\alpha}<1 \tag{13}$$

then the FBVP (1)-(2) has a unique solution on J.

Proof. We transform the problem (1)-(2) into a fixed point problem. Consider the operator

$$F(y)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s))+\int_0^s g\,(s,\tau,y(\tau))d\tau\right)ds$$
$$-\frac{b}{a+b}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s))+\int_0^s g\,(s,\tau,y(\tau))d\tau\right)ds$$
$$+\frac{c}{a+b} \tag{14}$$

Clearly, the fixed points of the operator F are solutions of the problem (1)-(2). We shall use the Banach contraction principle to prove that F defined by (14) has a fixed point. We shall show that F is a contraction.

Let $x,y\in C(J,\mathbb{R})$. Then, for each $t\in J$ we have

$$|F(x)(t)-F(y)(t)|$$
$$\le\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))-f(s,y(s))|ds$$
$$+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s|\,g(s,\tau,x(\tau))-g(s,\tau,y(\tau))|d\tau ds$$
$$+\frac{|b|}{|a+b|}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))-f(s,y(s))|ds$$
$$+\frac{|b|}{|a+b|}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s|\,g(s,\tau,x(\tau))-g(s,\tau,y(\tau))|d\tau ds$$

$$\le\frac{k\parallel x-y\parallel_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds$$
$$+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\beta-1}|h(s,x(s))-h(s,y(s))|d\tau ds$$
$$+\frac{k|b|\parallel x-y\parallel_\infty}{\Gamma(\alpha)|a+b|}\int_0^T(T-s)^{\alpha-1}ds$$

$$+\frac{|b|}{|a+b|}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\beta-1}|h(s,x(s))-h(s,y(s))|d\tau ds$$

$$\le\left[\frac{kT^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{k'T^{\alpha}T^{\beta}}{\alpha\beta\Gamma(\alpha)}+\frac{|b|(kT^{\alpha})}{\alpha\Gamma(\alpha)|a+b|}+\frac{|b|(k'T^{\alpha+\beta})}{\alpha\beta\Gamma(\alpha)|a+b|}\right]\parallel x-y\parallel_\infty$$

$$\le\frac{(k\beta+k'T^{\beta})|a+b|+(k+k')|b|T^{\beta}}{\alpha\beta\Gamma(\alpha)|a+b|}T^{\alpha}\parallel x-y\parallel_\infty$$

Consequently by (13) F is a contraction. As a consequence of Banach fixed point theorem, we deduce that F has an unique fixed point which is unique solution of the problem (1)-(2). □

he second result is based on Kransnoselskii's fixed point theorem.

Theorem 9. Assume (H1) in theorem 8 is hold and

(H3) There exists a constant $M>0$ such that

$|f(t,u)|\le M$, for each $t\in J$, and for all $u\in\mathbb{R}$ and

(H4) There exists a continuous function $h:[0.T]\times\mathbb{R}\to\mathbb{R}$ and $0<\beta<1$ such that $|g(t,s,u)|\le(t-s)^{\beta-1}|h(s,u)|$ and there exists $N>0$ such taht $|h(t,u)|\le N$ for all $t\in[0,T],u\in\mathbb{R}$

if

$$\frac{kT^{\alpha}}{\alpha\Gamma(\alpha)}\left(1+\frac{|b|}{|a+b|}\right)<1 \tag{15}$$

then the BVP (1)-(2) has solution on J.

Proof. We shall use Kransnoselskii's fixed point theorem to prove that F defined by (14) has a fixed point. We can write F

by $F(y)(t) = Ay(t) + By(t)$ such as

$$Ay(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,y(s))}{(t-s)^{1-\alpha}} - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T \frac{f(s,y(s))}{(T-s)^{1-\alpha}}ds - c\right]$$

and

$$By(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s g(s,\tau,y(\tau))d\tau ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\int_0^s g(s,\tau,y(\tau))d\tau ds\right]$$

The proof will be given in several steps.

Step1: F is an into mapping.

Let

$$r \geq \frac{(|a+b|+|b|)(M\beta + NT^{\beta})T^{\alpha}}{|a+b|\Gamma(\alpha+1)\beta} + \frac{c}{|a+b|}, \tag{16}$$

we define

$B_r = \{x \in C(J,\mathbb{R}) : \| x \|_\infty \leq r\}$.

It is clear that B_r is convex and closed set, consider defined mapping F by (14). By $(H4)$ we have

$$\begin{aligned}
&|F(y)(t)| \\
&\leq \sup_{t\in J}\left\{\int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha)}|f(s,y(s))|ds\right\} + \frac{|b|}{|a+b|}\sup_{t\in J}\left\{\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha)}|f(s,y(s))|ds\right\} + \frac{|c|}{|a+b|} \\
&+ \sup_{t\in J}\left\{\int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha)}\int_0^s |g(s,\tau,y(\tau))|d\tau ds\right\} \\
&+ \sup_{t\in J}\left\{\frac{|b|}{|a+b|}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha)}\int_0^s |g(s,\tau,y(\tau))|d\tau ds\right\} \\
&\leq \frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|}\frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} \\
&+ \sup_{t\in J}\left\{\int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\beta-1}|h(\tau,y(\tau))|d\tau ds\right\} \\
&+ \sup_{t\in J}\left\{\frac{|b|}{|a+b|}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\beta-1}|h(\tau,y(\tau))|d\tau ds\right\} \\
&\leq \frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|}\frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} \\
&+ \frac{MT^{\alpha+\beta}}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|}\frac{MT^{\alpha+\beta}}{\Gamma(\alpha+1)} = r
\end{aligned}$$

That is $(B_r) \subset B_r$.

Step2: A is contraction mapping.

Let $x, y \in C(J,\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned}
&|A(x)(t) - A(y)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds \\
&+ \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds \\
&\leq \frac{k \| x-y \|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds
\end{aligned}$$

$$+\frac{k|b| \parallel x-y \parallel_\infty}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}ds$$

$$\leq \frac{k}{\alpha\Gamma(\alpha)}T^\alpha \parallel x-y \parallel_\infty + \frac{|b|(k)}{\alpha\Gamma(\alpha)|a+b|}T^\alpha \parallel x-y \parallel_\infty$$
$$\leq \frac{kT^\alpha}{\alpha\Gamma(\alpha)}\left(1+\frac{|b|}{|a+b|}\right) \parallel x-y \parallel_\infty$$

Consequently by (15) A is a contraction.

Step3: B is continuous.
Let y_n be a sequence such that $y_n \to y$ in [Trial mode] Then for each [Trial mode]
[Trial mode]
Since [Trial mode] is a continuous function, we have
[Trial mode]
That is [Trial mode]

Step4: [Trial mode] maps bounded set into bounded set.
Indeed, it is enough to show that for any [Trial mode], there exists a positive constant [Trial mode] such that for each [Trial mode], we have [Trial mode].
By *(H4)* we have for each [Trial mode],
[Trial mode]

Step5: [Trial mode] maps bounded sets into equicontinuous sets of [Trial mode].
Let [Trial mode] and [Trial mode] be a bounded set of [Trial mode] as in Step 4, and let [Trial mode]. Then [Trial mode]
As [Trial mode], the right-hand side of the above inequality tends to zero.
From the uniform boundedness and the equicontinuity we deduce by the Arzela-Ascoli theorem that [Trial mode] is relatively compact. The Kransnoselskii's fixed point theorem assures the existence of at least one fixed point in [Trial mode] for the operator [Trial mode].
Remark 10. Our results for the FBVP (1)-(2) are applied for initial value problems [Trial mode], terminal value problems [Trial mode] and anti-periodic solutions [Trial mode].
Example 11. In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem, [Trial mode]Set [Trial mode]and [Trial mode]
Let [Trial mode] and [Trial mode] Then we have [Trial mode]Hence the condition [Trial mode] holds with [Trial mode]. We shall check that condition (15)is satisfied for appropriate values of [Trial mode] with [Trial mode]. Indeed
[Trial mode]and [Trial mode]
Then by Theorem 9 the problem (17)-(18) has solution on [Trial mode] for values of [Trial mode] satisfying condition (19).

Acknowledgments
The authors are extremely grateful to the referee for useful suggestions that improved the content of the paper.

References
A. A. Kilbas, H. M. Srivastava , J. J. Trujillo , Theory And Applications of Fractional Differential Equations North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006.
A. A. Kilbas, O. I. Marichev and S. G. Samko, Fractional Integral And Derivatives (Theory and Applications), Gordon and Breach, Switzerland, 1993.
A. A. Kilbas and S . A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, J. Diff. Equa. **41**(2005), 84-89.
Agarwal R. P, Benchohra M, Hamani S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta. Appl. Math. 2010; **109**: 973-â€"1033.
Ahmadkhanlu A, Jahanshahi M. On the Existence and Uniqueness of Solution of Initial Value problem for Fractional order Differential Equations on Time scales. Bull. Ira. Math. Soc. 2012; **38**(1): 241-252.
Bass H, Kingman J. F. C, Smithies F. Fixed Point Theoreams. Britain: Cambridge University Press, 1974.
Bayin S. Mathematical Methods in Science and Engineering. United States: John Wiley & Sons, Inc., 2006.
Benchohra M, Hamani S, Ntouyas S. K. Boundary value problems for differential equations with fractional order. Surveys in Mathematics and its Applications 2008;**3**: 1 â€" 12.
Diethelm K, Ford N. Analysis of fractional differential equations, J. Math. Anal. Appl. 2002; **265**: 229-248.
Douglas J. F. Some applications of fractional calculus to polymer science, Advances in chemical physics. John Wiley & Sons, Inc. 1997; **102**: 121-191.
El-Sayed A. M. A, Ibrahim A. G. Set valued integral equations of fractional orders. Appl. Math. Comput. 2001; **118**; 113-121.
F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena , Chaos, Solitons and Fractals, **7**(9) 146-1477, (1996)
Fellah Z. E. A, Depollier C. Application of fractional calculus to the sound waves propagation in rigid porous materials:Validation via ultrasonic measurement., Acta Acustica 2002; **88**: 34-39.
Furati K. M, Tatar N. An Existence Result for a Nonlocal Fractional Differential Problem. J. Frac. Calculus. 2004; **26**: 43-51.
Hadid S. B, Masaedeh B, Momani S. , On the exitence of maximal and minimal solutions of differential equations of non-integer order. J. Fract. Calculus 1996; **9**: 41-44.

Hadid S. B. Local and global exitence theorems on differential equations of non-integer order., J. Fract. Calculus 1995; **7**: 101-105.
I. Podlubny, Fractional Dï¬€erential Equations Academic Press,1999.
M. Jahanshahi, A. Ahmadkhanlu, N. Aliev, On Well-Posed of Boundary Value Problems Including Fractional Order Differential Equation Bull. Sou. Asi. Math. Soc. In Press No. 37.
R. Hilfer, Applications of Fractional Calculus in Physics,WorldScientific Publishing Co. 2000
S. M. Momani, Local and global uniqueness theorems on differential equations of non-integer order via Bihariâ€™s and Gronwallâ€™s inequalities, Revista Tecnica J.**23**(2) (2000) 66-69.
V. A. Plotnikov, A. V. Plotnikov and A. N. Vityuk, Diferential Equations with a Multivalued Right-Hand Side, Asymptotic Methods "AstroPrint", Odessa, 1999.